Introduction to Choquet calculus

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In this talk we give a brief introduction to a recent topic "Choquet calculus" where calculus consists of integrals and derivatives.

Modern integrals, namely Lebesgue integrals initiated by Lebesgue in 1902, are associated with the concept of "measures". Lebesgue measures are defined as additive set functions with certain conditions and, hence, Lebesgue integrals hold additivity by inheriting the property of measures. In the latter half of the 20th century, a new concept of "measures without additivity" named fuzzy measures was proposed by Sugeno in 1974. Fuzzy measures (non-additive measures in general) are defined as monotone set functions and considered as a natural extension of Lebesgue measures leading to the concept of non-additive integrals with respect to non-additive measures, where we note that monotonicity contains additivity in it.

In 1953, Choquet studied the so-called Choquet functionals based on capacities where capacities representing "potential energy" of a set were also monotone set functions but not captured as "measures" as in the sense of Lebesgue. Together with the appearance of fuzzy measures, Choquet functionals were finally re-formulated as non-additive integrals with respect to fuzzy measures by Höle in 1982. Since then, various non-additive integrals with respect to non-additive measures have been suggested. Among them, we focus on Choquet integrals which are the most general extension of Lebesgue integrals

Once we obtain the concept of integrals, we become curious about their inverse operations. In the case of Lebesgue integrals, Radon and Nikodym gave Radon-Nikodym derivatives as inverse operations in 1913 and 1930, respectively. It is well-known that with the aid of Radon-Nikodym derivatives, Kolmogorov proved the existence of conditional probabilities in 1933 and thus initiated modern probability theory where probabilities are nothing but Lebesgue measures. On the other hand in fuzzy measure theory, conditional fuzzy measures have been not well defined in the sense of Kolmogorov. Very recently, inverse operations of Choquet integrals were studied as "derivatives with respect to fuzzy measures" (Sugeno 2013).

In this talk, we deal with Choquet calculus (Sugeno 2015) based on Choquet integrals and derivatives. So far most studies on Choquet integrals have been devoted to a discrete case. In Choquet calculus we deal with continuous Choquet integrals and derivatives as well. First we show how to calculate continuous Choquet integrals. To this aim, we consider distorted Lebesgue measures as a special class of fuzzy measures, and non-negative and non-decreasing functions. The Laplace transformation is used as a basic tool for calculations. Distorted Lebesgue measures are obtained by monotone transformations of Lebesgue measures according to the idea of distorted probabilities suggested by Edwards in 1953. We remember that Kahneman was awarded Nobel Prize in Economics in 2002 where his cumulative prospect theory is based on "Choquet integrals with respect to distorted probabilities". Next we define derivatives of functions with respect to distorted Lebesgue measures where the derivatives correspond to Radon-Nikodym derivatives in the case of Lebesgue integrals. We also discuss the identification of distorted Lebesgue measures which is a problem arising particularly in Choquet calculus. Then we show some relations between Choquet calculus and fractional calculus which is recently getting very popular, for example, in control theory. Also we consider differential equations with respect to distorted Lebesgue measures and give their solutions. Lastly we present the concept of conditional distorted Lebesgue measures defined with the aide of Radon-Nikodym-like derivatives.